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AN EFFICIENT STRESS FUNCTION APPROXIMATION FOR THE FREE-EDGE STRESSES IN LAMINATES

GERRY FLANAGAN
Grumman Aerospace and Electronics, Mail Stop B44-35, Bethpage, NY 11714, U.S.A.

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Abstract—A solution method is derived for determining the free-edge stresses in composite laminates. The method is based on expanding stress functions in terms of a harmonic series in the thickness direction. Using the principle of minimum complementary energy, a system of ordinary differential equations is derived for the distribution in the width direction. Cross-ply and angle-ply laminates are examined to establish the validity of the solution. Eight- and 16-ply quasi-isotropic laminates are also examined to demonstrate a convergence criterion based on average stress. The method proves to be a relatively simple and efficient approach for this problem.

1. INTRODUCTION

Determination of the free-edge stress is an important step in preventing delamination from initiating at the free edge. Unfortunately, closed-form solutions for the stresses do not exist, and the steep stress gradients that typically occur in these regions challenge conventional numerical methods. Although free-edges can occur in geometrically complex situations, most researchers have concentrated on the case of a long, uniform laminate, subjected to a uniaxial mechanical strain, as shown in Fig. 1. This specialized form of the problem will be addressed in the current analysis, with residual thermal stresses also considered.

From an engineering viewpoint, an analysis method is desired which can be applied to structural laminates. Structural laminates will include a mixture of cross-plies (0° and 90°) and angle plies, and the total number of plies may be large. A candidate analysis method should require a minimum of input data (and thus analyst's time) and computer resources. It should yield the full stress field in the region near the edge, or at least the complete stress state at each ply interface. Finally, there must be some method of establishing convergence of the stresses in terms of some error metric. Because a closed-form solution to this problem cannot be obtained, a series solution (with either continuous or piece-wise basis functions) of some type must be used. Achieving an arbitrary degree of convergence with a series solution requires that the number of degrees-of-freedom in the approximation can be arbitrarily increased.

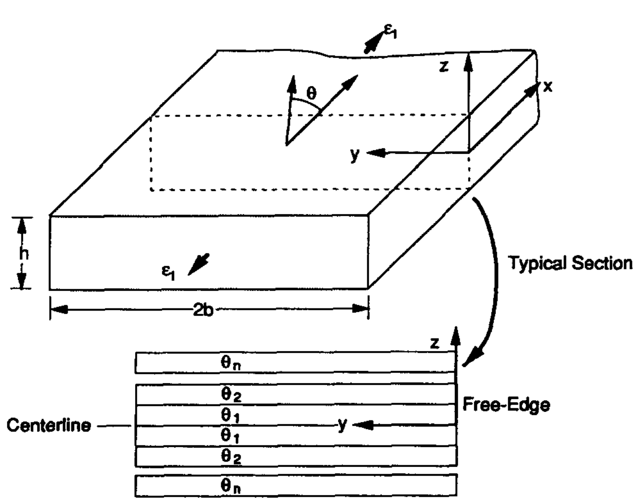

Fig. 1. Geometry and coordinate system for laminate free-edge problem.

The analysis of the stress-state near the free-edge of a laminate has generated a large body of literature. A review of the early literature on the subject is given by Salamon (1980). A more recent review is given by Sandhu *et al.* (1991). Several distinct approaches for finding approximate solutions for the stress distribution have been suggested. These include displacement-based finite difference (Pipes *et al.*, 1970; Atlus *et al.*, 1980) and finite element (Wang and Crossman, 1977; Whitcomb and Raju, 1983, 1984) methods. Convergence of these methods is slow, requiring fine grids and thus large computer resources to resolve the stresses. Spilker (1980) demonstrated the importance of exactly satisfying the traction-free boundary conditions at the edge in order to ensure convergence. The advantages to be gained by explicitly satisfying these conditions has led to a variety of mixed formulation (Pagano, 1978) and stress-based methods. Rybicki (1971) used an equilibrium finite element analysis in which a stress function was described by a series of Hermitian interpolation polynomials within each discrete element. Tang (1975) used a power series expansion in the thickness coordinate of the stress function. Substituting terms of the series into the compatibility equations leads to a set of ordinary differential equations in the width coordinate. The present method builds on this approach, but will be formulated in a way that allows one to analyze more general stacking sequences and establish convergence. In addition, Tang satisfied the boundary conditions only in an average sense. Bar-Yoseph and Pian (1981) expanded the stresses using Legendre polynomials in the thickness direction. By applying the principle of minimum complimentary energy, a set of ordinary differential equations in the width coordinate was obtained. The formulation by Bar-Yoseph and Pian was specialized for the case of a $[\pm\theta]$ laminate, and it is not apparent how to expand the approach into a general analysis tool. Kassapoglou and Lagace (1986) determined distributions for the stress functions based on integrated force and moment equilibrium. The resulting algorithm, referred to as the force-balance method, is efficient, with solution time increasing slowly with the number of plies in the laminate. Unfortunately, the approach leads to the assumption that the in-plane stresses are constant through the thickness of a given ply. A solution to an elasticity problem should not impose artificial restrictions on the stress distributions unless it can be established that the restrictions have a negligible effect on the results. It will be shown later in this paper that the assumption of constant in-plane stresses leads to interlaminar stress distributions that are substantially different from those obtained by less restrictive methods.

Yin (1991) approximated the stress functions using piecewise polynomial approximations in the thickness coordinate. In this approach, the stress functions are continuous over a layer, and continuity of the interlaminar stress is enforced at the layer interfaces. This method also leads to a system of ordinary differential equations in the width coordinate. Although the analysis given by Yin has most of the desired features (general laminate can be analyzed, full-field solution for the stresses are available and convergence could be established by subdividing layers), it is desirable to have a simpler method available.

The singular nature of the stresses at a ply interface was studied by Wang and Choi (1982a). Once the form of the singularity had been established, a detailed description of the local stress field was possible (Wang and Choi, 1982b). The present method does not include a singular stress field. Kassapoglou and Lagace (1986) and Wang and Crossman (1977) have discussed the utility of solutions that do not include the singularity. The singularities arise from assuming a ply is a homogeneous orthotropic body. In reality, the ply is heterogeneous (fibers and matrix) at the same dimensional scale at which the singularity is important. Therefore, useful engineering information can be obtained from methods which do not include this feature.

2. THEORETICAL APPROACH

In the solution scheme that follows, approximate series expressions for the stresses within a composite laminate will be obtained. Stress functions are used to ensure that stress equilibrium is satisfied everywhere. A variational technique based on the complimentary strain energy is applied so that compatibility will be satisfied in an integral sense. The stress functions are assumed to be separable functions in the y and z coordinates. The z function is an assumed series, while the y functions are determined by satisfying ordinary differential equations that result from application of the variational method. All traction boundary conditions will be satisfied exactly. Classical laminate theory is approximated in the interior of the laminate. By taking a sufficiently large number of terms in the z direction, compatibility can be approximated to the desired level of accuracy. The basic theory is applicable to finite width laminates with arbitrary stacking sequences. However, for simplicity, the solution has been specialized for wide laminates (width $\gg$ thickness), with symmetric stacking sequences.

Consider a composite laminate with a traction-free edge, as shown in Fig. 1. The laminate consists of layers of uniform thickness orthotropic material. Each layer may be oriented at an arbitrary angle in the x–y plane. In the laminate coordinate system, the constitutive equations for an individual layer may be expressed by the generalized Hooke's law in contracted notation as

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & S_{16} \\ S_{12} & S_{22} & S_{23} & 0 & 0 & S_{26} \\ S_{13} & S_{23} & S_{33} & 0 & 0 & S_{36} \\ 0 & 0 & 0 & S_{44} & S_{45} & 0 \\ 0 & 0 & 0 & S_{45} & S_{55} & 0 \\ S_{16} & S_{26} & S_{36} & 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} + \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ 0 \\ 0 \\ \alpha_6 \end{Bmatrix} \Delta T, \tag{1}$$

where S_{ij} are terms of the transformed compliance matrix, α_i are the transformed thermal expansion coefficients and ΔT is the temperature change from a reference, stress-free condition.

The stress boundary conditions for the problem are as follows:

$$\begin{aligned} &\sigma_2 = \sigma_4 = \sigma_6 = 0 \quad \text{at } y = 0, b \\ &\sigma_3 = \sigma_4 = \sigma_5 = 0 \quad \text{at } z = \pm h/2. \end{aligned} \tag{2}$$

The laminate is assumed to be in a state of plane strain in the x direction. The concept of plane strain is generalized to allow for a specified value of ε_1. Under these assumptions, the axial stress can be found from

$$\sigma_1 = (\varepsilon_1 - \alpha_1 \Delta T - S_{1j}\sigma_j)/S_{11} \quad (j = 2, 3, 6). \tag{3}$$

In (3), and the equations that follow, tensor summation rules are assumed. The constant value of ε_1 can be reflected in the generalized Hooke's law by rearranging (1) into the following form:

$$\varepsilon_i = \hat{S}_{ij}\sigma_j + \frac{S_{i1}}{S_{11}}\varepsilon_1 + \hat{\alpha}_i \Delta T \quad (i, j = 2, 3, \ldots, 6), \tag{4}$$

where

$$\hat{S}_{ij} = S_{ij} - \frac{S_{i1}S_{1j}}{S_{11}}, \quad \hat{\alpha}_i = \alpha_i - \frac{S_{1i}}{S_{11}}\alpha_1. \tag{5}$$

Two nondimensional coordinates are introduced:

$$\eta = z/h, \quad \xi = y/h. \tag{6}$$

The Lekhnitskii stress functions (Lekhnitskii, 1968) are used to ensure pointwise equilibrium. These are defined by

$$\frac{\partial^2 F}{\partial \eta^2} = \sigma_2 \qquad \frac{\partial^2 F}{\partial \xi^2} = \sigma_3 \qquad \frac{\partial^2 F}{\partial \xi\, \partial \eta} = -\sigma_4$$

$$\frac{\partial \Psi}{\partial \xi} = -\sigma_5 \qquad \frac{\partial \Psi}{\partial \eta} = \sigma_6. \tag{7}$$

We will approximate the stress functions using the following series

$$F = \sum_{i=1}^{n} f_i(\xi) g_i(\eta)$$

$$\Psi = \sum_{i=1}^{n} p_i(\xi) g_i^{\mathrm{I}}(\eta). \tag{8}$$

The superscript Roman numerical gives the order of differentiation of the function with respect to the function's argument. Although the series for F and Ψ can be defined independently, using the derivative of $g(\eta)$ from the F series as the η function in the Ψ series will lead to certain simplifications in later expressions. In the development that follows, we will assume that $g(\eta)$ is a known function that satisfies the stress boundary conditions for the problem at $\eta = \pm 1/2$.

The governing equations are obtained by taking the stationary value of the complementary strain energy

$$\partial U = \iint \varepsilon \delta \sigma \, \mathrm{d}y \, \mathrm{d}z$$

$$= \iint [\sigma_i \hat{S}_{ij} \delta \sigma_j + (\varepsilon_1 S_{i1}/S_{11} + \hat{\alpha}_i) \delta \sigma_j] \, \mathrm{d}\xi \, \mathrm{d}\eta = 0 \quad (i, j = 2, 3, \ldots, 6). \tag{9}$$

Next, the stress functions (7) and their series expansions (8) are substituted into (9). Repeated integration by parts is then applied to obtain

$$\int [a_{ij}^{(4)} f_j^{\mathrm{IV}} + a_{ij}^{(2)} f_j^{\mathrm{II}} + a_{ij}^{(0)} f_j + b_{ij}^{(2)} p_j^{\mathrm{II}} + b_{ij}^{(0)} p_{ij} + r_j] \delta f_i \, \mathrm{d}\xi$$

$$+ \int [c_{ij}^{(2)} p_j^{\mathrm{II}} + c_{ij}^{(0)} p_j + b_{ij}^{(2)} f_{ij}^{\mathrm{II}} + b_{ij}^{(0)} f_{ij} + s_j] \delta p_i \, \mathrm{d}\xi = 0 \quad (i, j = 1, 2, \ldots, n), \tag{10}$$

where

$$a_{ij}^{(4)} = \int_{-1/2}^{1/2} \hat{S}_{33} g_i g_j \, \mathrm{d}\eta \qquad b_{ij}^{(2)} = -\int_{-1/2}^{1/2} (\hat{S}_{36} + \hat{S}_{45}) g_i^{\mathrm{I}} g_j^{\mathrm{I}} \, \mathrm{d}\eta$$

$$a_{ij}^{(2)} = -\int_{-1/2}^{1/2} (2\hat{S}_{23} + \hat{S}_{44}) g_i^{\mathrm{I}} g_j^{\mathrm{I}} \, \mathrm{d}\eta \qquad b_{ij}^{(0)} = \int_{-1/2}^{1/2} \hat{S}_{26} g_i^{\mathrm{II}} g_j^{\mathrm{II}} \, \mathrm{d}\eta$$

$$a_{ij}^{(0)} = \int_{-1/2}^{1/2} \hat{S}_{22} g_i^{\mathrm{II}} g_j^{\mathrm{II}} \, \mathrm{d}\eta \qquad r_j = \int_{-1/2}^{1/2} (\varepsilon_1 S_{12}/S_{11} + \hat{\alpha}_2) g_j^{\mathrm{II}} \, \mathrm{d}\eta$$

$$c_{ij}^{(2)} = -\int_{-1/2}^{1/2} \hat{S}_{55} g_i^{\mathrm{I}} g_j^{\mathrm{I}} \, \mathrm{d}\eta \qquad s_j = \int_{-1/2}^{1/2} (\varepsilon_1 S_{16}/S_{11} + \hat{\alpha}_6) g_j^{\mathrm{II}} \, \mathrm{d}\eta$$

$$c_{ij}^{(0)} = \int_{-1/2}^{1/2} \hat{S}_{66} g_i^{\mathrm{II}} g_j^{\mathrm{II}} \, \mathrm{d}\eta. \tag{11}$$

In taking these integrations, it is recognized that the $\hat{S}_{ij}$ terms are stepwise functions of z

that depend on the local layer material properties and orientation. The boundary condition terms generated by the integration by parts vanish because we have assumed that $g(\eta)$ identically satisfies the traction-free conditions on the laminate surfaces.

Because δf and δp are arbitrary variations, the quantities inside the square brackets must vanish for eqn (10) to be satisfied. Thus, eqn (10) leads to a series of coupled, ordinary differential equations in $f_i(\xi)$ and $p_i(\xi)$. Solutions for the homogeneous system are of the form

$$f_i = v_i^f \mathrm{e}^{\alpha\xi}, \quad p_i = v_i^p \mathrm{e}^{\alpha\xi}. \tag{12}$$

Substituting (12) into (10) results in the following system of linear algebraic equations:

$$\begin{aligned} a_{ij}^{(0)} v_j^f + (a_{ij}^{(2)} + \alpha^2 a_{ij}^{(4)}) v_j^{f''} + (b_{ij}^{(0)} + \alpha^2 b_{ij}^{(2)}) v_j^p &= 0 \\ b_{ij}^{(0)} v_j^f + b_{ij}^{(2)} v_j^{f''} + (c_{ij}^{(0)} + \alpha^2 c_{ij}^{(2)}) v_j^p &= 0 \\ \alpha^2 v_j^f - v_j^{f''} &= 0 \quad (i, j = 1, 2, \ldots, n). \end{aligned} \tag{13}$$

The last expression in (13) comes from introducing a dummy variable such that

$$f_i'' = v_i^{f''} \mathrm{e}^{\alpha\xi}. \tag{14}$$

Equation (13) can be solved as an eigensystem, with α^2 as the eigenvalue and the v coefficients forming the eigenvector. To simplify the remaining derivation, we assume that the y dimension of the laminate is large compared to the z dimension. The origin of the ξ coordinate shown in Fig. 1 is located at the free edge of the laminate. Thus, we are seeking functions that decay with increasing ξ. This is conveniently done by taking only the negative roots of α^2. Equation (13) results in $3n$ eigenvalues. The complete homogeneous solution is given by the linear superposition of all $3n$ terms as follows:

$$\begin{aligned} f_i^{(\mathrm{H})} &= v_{ij}^f t_j \mathrm{e}^{-\alpha_j\xi} \\ p_i^{(\mathrm{H})} &= v_{ij}^p t_j \mathrm{e}^{-\alpha_j\xi} \quad (i = 1, 2, \ldots, n), (j = 1, 2, \ldots, 3n), \end{aligned} \tag{15}$$

where t_j are constants which will be determined by the boundary conditions.

A particular solution can be constructed by assuming $f_i(\xi)$ and $p_i(\xi)$ are constants. These constants are found by solving the following linear system of equations:

$$\begin{aligned} a_{ij}^{(0)} f_j^{(P)} + b_{ij}^{(0)} p_j^{(P)} &= -r_i \\ b_{ij}^{(0)} f_j^{(P)} + c_{ij}^{(0)} p_j^{(P)} &= -s_i \quad (i, j = 1, 2, \ldots, n). \end{aligned} \tag{16}$$

The particular solution can be identifed as a series approximation to the classical lamination theory stresses. The total solution is then

$$\begin{aligned} f_i &= f_i^{(H)} + f_i^{(P)} \\ p_i &= p_i^{(H)} + p_i^{(P)} \quad (i = 1, 2, \ldots, n). \end{aligned} \tag{17}$$

The final step is to satisfy the boundary conditions $\sigma_2 = \sigma_4 = \sigma_6 = 0$ at $y = 0$. These conditions are met for all η by solving the following system of equations for t_j:

$$\begin{aligned} v_{ij}^f t_j &= -f_i^{(P)} \\ v_{ij}^f \alpha_i t_j &= 0 \\ v_{ij}^p t_j &= -p_i^{(P)} \quad (i = 1, 2, \ldots, n), (j = 1, 2, \ldots, 3n). \end{aligned} \tag{18}$$

An alternative solution procedure is suggested by Tang (1975). Instead of finding a particular solution in terms of the series approximation, classical lamination theory is assumed in the interior of the laminate. Then, instead of (18), the boundary conditions are found by equating the left-hand side of (18) to weighted integrals of the lamination theory stresses, taking advantage of the orthogonality of the assumed series. This approach satisfies compatability in the laminate interior, but satisfies the boundary conditions only in an average sense.

Finally, from (7), the stress components are given by the following double summations:

$$\begin{aligned}
\sigma_2 &= t_j v_{ij}^f e^{-\alpha_j \xi} g_i^{11}(\eta) + f_i^{(P)} g_i^{11}(\eta) \\
\sigma_3 &= \alpha_j^2 t_j v_{ij}^f e^{-\alpha_j \xi} g_i(\eta) \\
\sigma_4 &= -\alpha_j t_j v_{ij}^f e^{-\alpha_j \xi} g_i^1(\eta) \\
\sigma_5 &= -\alpha_j t_j v_{ij}^p e^{-\alpha_j \xi} g_i^1(\eta) \\
\sigma_6 &= t_j v_{ij}^p e^{-\alpha_j \xi} g_i^{11}(\eta) + p_i^{(P)} g_i^{11}(\eta) \quad (i = 1, 2, \ldots, n), (j = 1, 2, \ldots, 3n).
\end{aligned} \tag{19}$$

An admissible series of functions for $g(\eta)$ must have the following characteristics. First, the functions and their first derivatives must equal zero at $\eta = \pm 1/2$ in order to satisfy the stress boundary conditions. Second, the series must be complete in order to ensure convergence. Finally, it is desirable that the series be orthogonal. One choice for the series is suggested by the solution for the free vibration of a clamped–clamped beam. The solution to a vibration problem is naturally orthogonal, and the clamped–clamped boundary conditions meet the stated requirements at the endpoints. For symmetric laminates, only the even functions from the complete series need to be included. These are

$$g_i(\eta) = \cos(\beta_i \eta) + k_i \cosh(\beta_i \eta), \tag{20}$$

where

$$k_i = -\cos(\beta_i/2)\,\mathrm{sech}\,(\beta_i/2) \tag{21}$$

and β_i are given by the roots of the equation

$$\cosh(\beta/2)\sin(\beta/2) + \cos(\beta/2)\sinh(\beta/2) = 0. \tag{22}$$

Equation (20) can be shown to be orthogonal. Similar functions can be constructed for the antisymmetric and general laminate cases. Care must be taken in numerically evaluating these functions to avoid problems with overflow or round-off.

3. EXAMPLES

3.1. $[0/90]_S$ *laminate*

The $[0/90]_S$ laminate is a reference case examined in many previous studies. For comparison purposes, many of the results in the literature are based on the following assumed material properties for graphite/epoxy:

$$\begin{aligned}
E_x &= 20 \times 10^6 \text{ psi}, \quad E_y = E_x = 2.1 \times 10^6 \text{ psi} \\
G_{xy} &= G_{yz} = G_{xz} = 0.85 \times 10^6 \text{ psi} \\
\nu_{xy} &= \nu_{xz} = \nu_{yz} = 0.21 \\
\alpha_x &= 0.22 \times 10^{-6} \text{ in/in/°F}, \quad \alpha_y = 15.2 \times 10^{-6} \text{ in/in/°F},
\end{aligned}$$

where x, y and z refer to the fiber, transverse and thickness directions, respectively. α_x and α_y are thermal expansion coefficients. These properties were used in all of the following

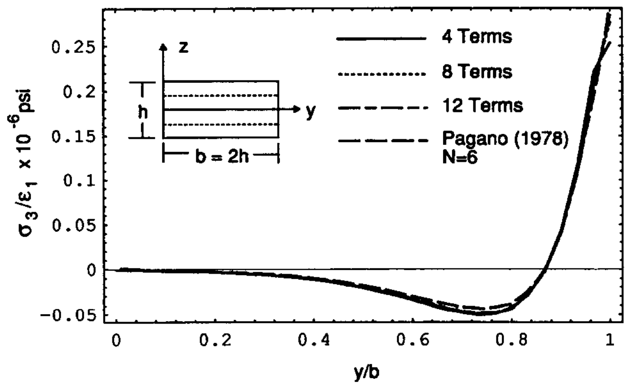

Fig. 2. σ_3 along centerline of $[0/90]_S$ laminate ($z/h = 0.0$).

example cases. However, experimental data indicate that $G_{yz} \neq G_{xz}$ and $v_{yz} \neq v_{xz}$ (Knight, 1982) for real graphite/epoxy.

Figure 2 shows the distribution of σ_3 stress at the midplane of the laminate for an applied axial strain. Three different values were used for the number of terms in the series ($n = 4, 8, 12$). The results rapidly converge to a constant value that is in excellent agreement with previously reported results as n increases. Spilker (1980), for example, reported a peak value for σ_3 at this interface of 0.285 psi. Results from Pagano (1978) are also plotted for comparison. In Pagano's work, the parameter N refers to the number of sublayers used to model one-half of the laminate. For plots of stress versus the y coordinate, we have changed the coordinate system origin and normalization to match the scale used in earlier papers. In this system, the origin of y lies eight ply thicknesses from the edge, and y is normalized by a width equal to eight ply thicknesses. The σ_4 component at the 0/90 interface is shown in Fig. 3. Here, the singular nature of the stresses at the material interface is evident, with the peak stress steadily increasing as the number of terms increases.

More complex behavior is evident when the σ_3 component is plotted for $y = 0$ (exactly at the free edge), as shown in Fig. 4. There is a peak value of stress that occurs near the 0/90 interface that appears to be increasing without limit as the number of terms increases, reflecting the singular nature of the stresses at that point. The harmonic series representation of this sudden peak causes oscillations to appear elsewhere in the distribution, much as a Fourier series oscillates when approximating a sudden discontinuity in a function. As will be shown below in the discussion of quasi-isotropic laminates, these oscillations decay rapidly as y increases and are not a factor when an average stress criterion is used. Figure 4 compares favorably with similar plots given by Sandu *et al.* (1991), Wang and Crossman (1977), and Spilker (1980). Results from Sandu's work are reproduced in Fig. 4. These results were generated with a finite element model which used 288 continuous traction

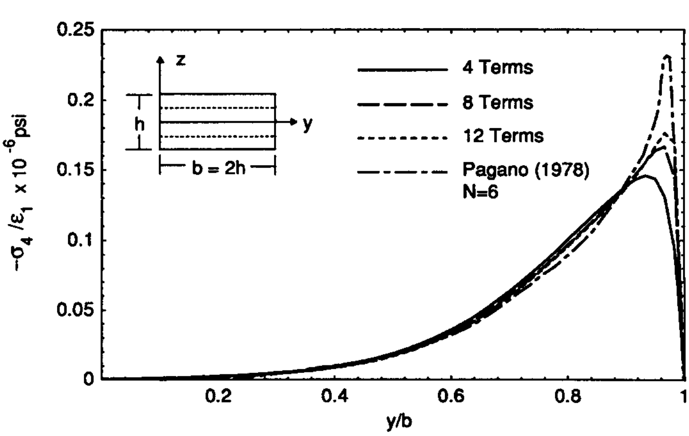

Fig. 3. σ_4 along 0/90 interface of $[0/90]_S$ laminate ($z/h = 0.25$).

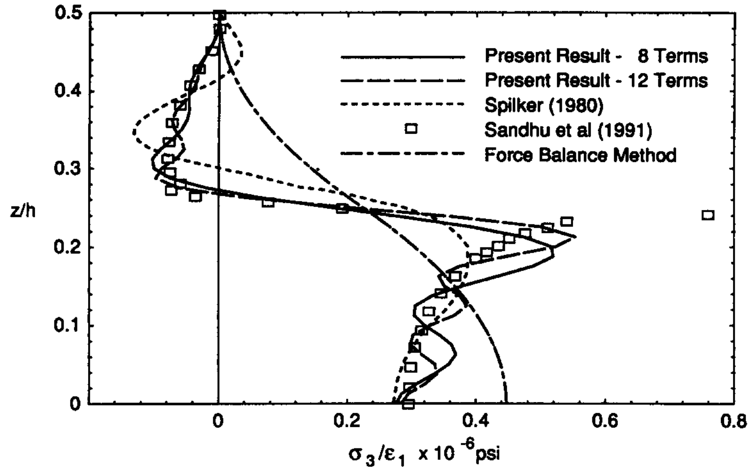

Fig. 4. σ_3 at free edge ($y = 0$) of $[0/90]_S$ laminate.

elements. Also shown are results obtained from the force-balance method by Kassapoglou and Lagace (1986). This method gives a rough approximation for the midplane stress ($z = 0$), but cannot match the details of the distribution obtained by using less restrictive assumptions.

3.2. $[45/-45]_S$ *laminate*

The $[45/-45]_S$ laminate is another standard case examined by several researchers. The σ_5 component at the $45/-45$ interface is given in Fig. 5. The peak value seems to be increasing without limit as the number of terms increases, but the distribution immediately beyond the edge converges rapidly. Results from Pagano (1978) are shown for comparison. The distributions for σ_3 and σ_4 are shown in Fig. 6 (computed for $n = 12$). These distributions can be compared with those given by Sandu *et al.* (1991) and Wang and Choi (1982). The distribution of the σ_3 component has the same sign and shape as that given by Wang and Choi, but the peak value is substantially less (0.17 versus 0.4).

3.3. *Quasi-isotropic laminates*

The quasi-isotropic laminates will be used to demonstrate the application of the method to practical laminates, and to address the issue of convergence. Clearly, an approximate

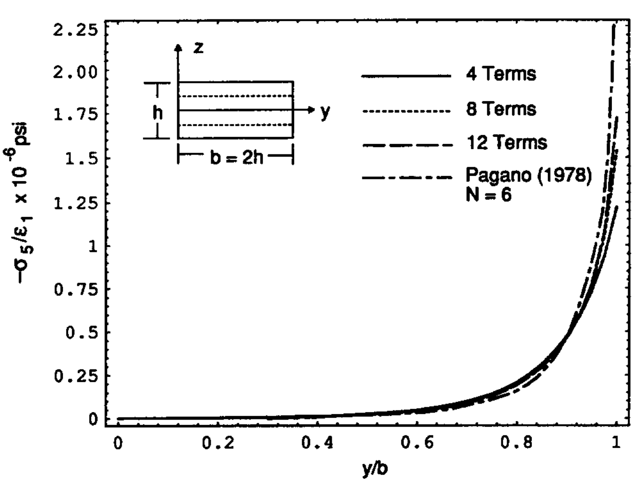

Fig. 5. σ_5 along $45/-45$ interface of $[45/-45]_S$ laminate ($z/h = 0.25$).

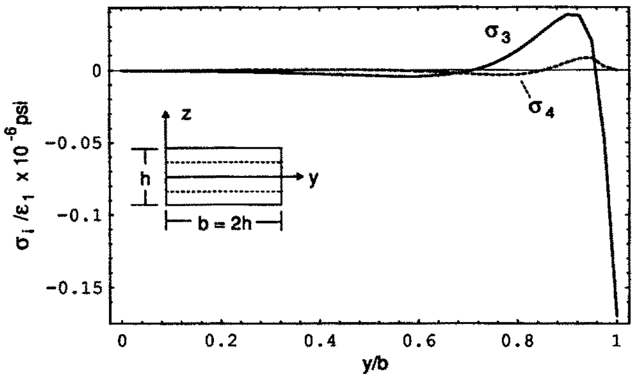

Fig. 6. σ_3 and σ_4 along 45/−45 interface of $[45/-45]_S$ laminate ($z/h = 0.25$).

solution such as this that does not represent the stress singularities cannot be expected to converge exactly in any meaningful way at the free edge. Kim and Soni (1984, 1986) have suggested a delamination initiation criterion based on the average stress taken over an interval starting at the edge and going to an empirically determined distance from the edge. An integration distance of one ply thickness was used by Kim and Soni. Thus, a practical measure of convergence is to examine how the average stress components approach a constant value as n increases.

The average σ_3 stress for a $[45/-45/0/90]_S$ laminate is shown in Fig. 7 for $n = 4$, 8 and 12. The plot shows that the stresses for $n = 9$ and $n = 12$ are nearly identical. A similar plot for the σ_5 component is given in Fig. 8. The averaging distance was equal to one ply thickness. For this case, the force-balance method is in good agreement with the present method.

The present solution method sacrifices an exact match to lamination theory in the interior of the laminate in order to satisfy the free-edge boundary conditions exactly. Figure 9 shows how the particular part of the series solution ($n = 12$) approaches the lamination theory result for the eight-ply quasi-isotropic laminate far from the edge. With 12 terms, the series approximation can only roughly approximate the stepwise changes in the interior in-plane stresses. However, the continuous interlaminar stresses, which are the problem of interest, converge much faster.

The average σ_3 stress for a $[45/-45/0/90]_{2S}$ laminate is shown in Fig. 10 for $n = 12$, 16 and 20. The plot shows that the stresses for $n = 16$ and $n = 20$ are nearly identical. These limited results indicate that the number of terms needs to be approximately equal to the

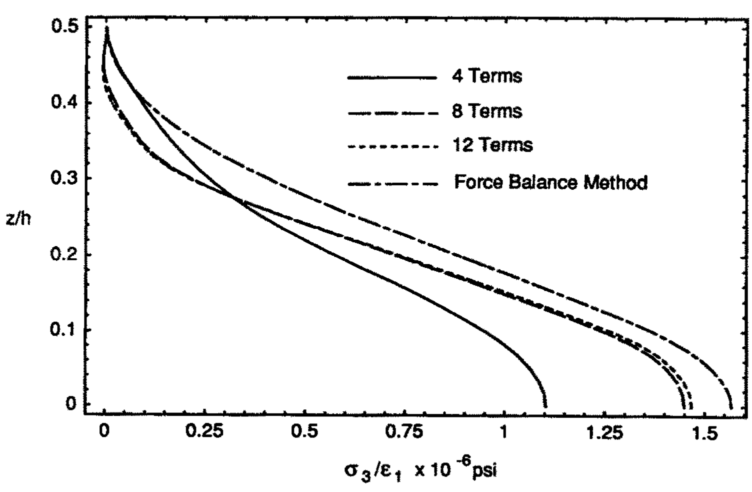

Fig. 7. Average σ_3 taken over interval $y/h = 0$ to $y/h = 1/8$ for $[45/-45/0/90]_S$ laminate.

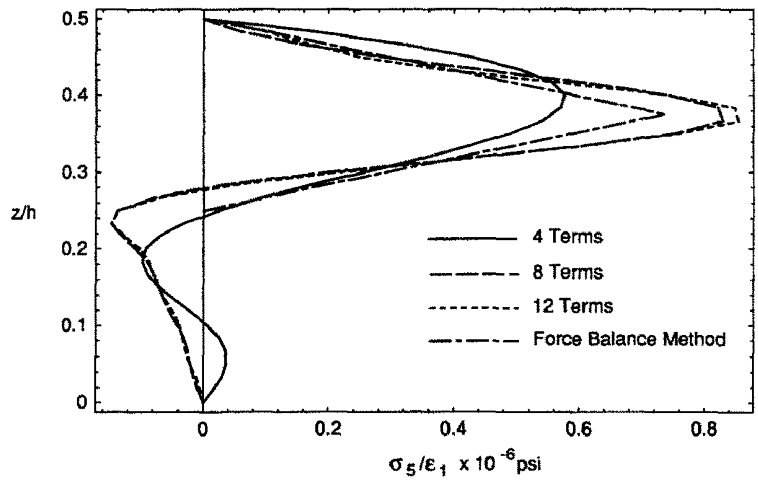

Fig. 8. Average σ_5 taken over interval $y/h = 0$ to $y/h = 1/8$ for $[45/-45/0/90]_S$ laminate.

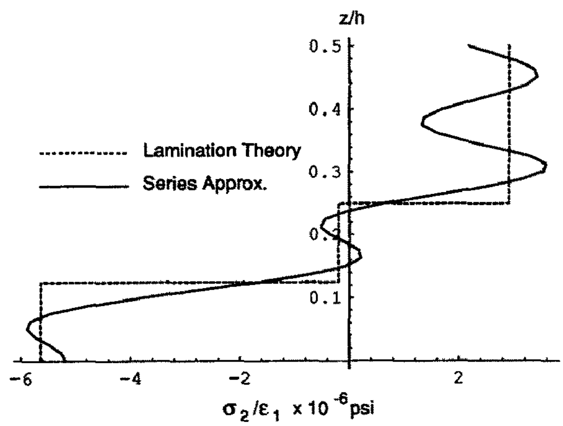

Fig. 9. σ_2 calculated from present solution and classical lamination theory at interior of laminate ($y/h = \infty$) for $[45/-45/0/90]_S$ laminate; $n = 8$.

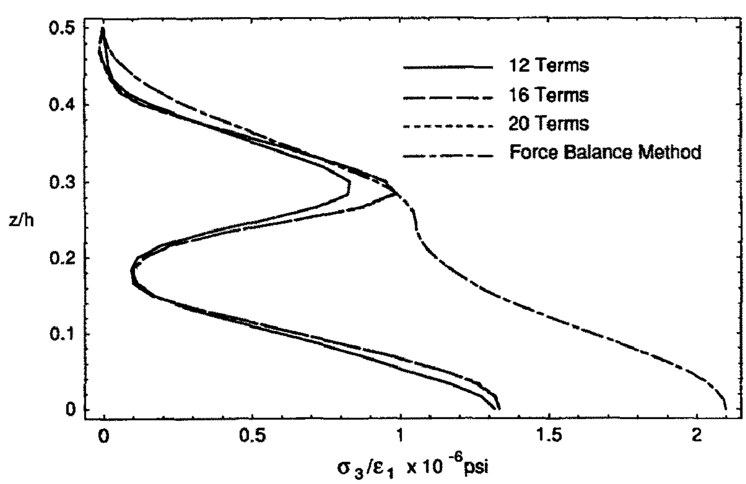

Fig. 10. Average σ_3 taken over interval $y/h = 0$ to $y/h = 1/16$ for $[45/-45/0/90]_{2S}$ laminate.

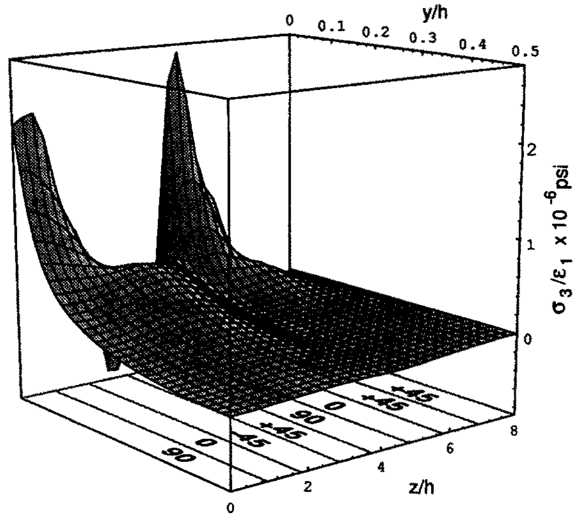

Fig. 11. σ_3 for upper half of $[45/-45/0/90]_{2S}$ laminate; $n = 16$.

number of plies for the average stresses to converge to constant values. For this repeating stacking sequence, the force-balance method gives a different stress distribution. A repeating stack should result in a stress distribution that repeats to a first approximation (Whitcomb, 1984), but the assumptions built into the force-balance method preclude this behavior.

The complexity of the full σ_3 distribution as obtained by the present method is shown in Fig. 11. Sixteen terms were used to create this plot.

In order to predict delamination initiation, the residual stresses resulting from cure must also be considered. Figure 12 shows the distribution of average σ_3 stress for the $[45/-45/90/0]_{2S}$ laminate due to residual thermal stress alone. For this laminate, the location of the peaks coincides with the peaks for the mechanically loaded case. In addition, the magnitude of the peaks are a significant fraction of the matrix strength (approximately 7.5 ksi). Thus, thermal effects would need to be included in a failure prediction.

4. CONCLUSIONS

A relatively simple and efficient method has been demonstrated for determining the stresses near the free edge of general composite laminates. Based on an average stress convergence criterion, a 16-ply laminate was analyzed using 16 terms in the series. In the solution procedure, a 16-term series requires that the eigenvalues of a 48×48 matrix must be determined, along with a 48×48 system of linear equations. Thus, the computational

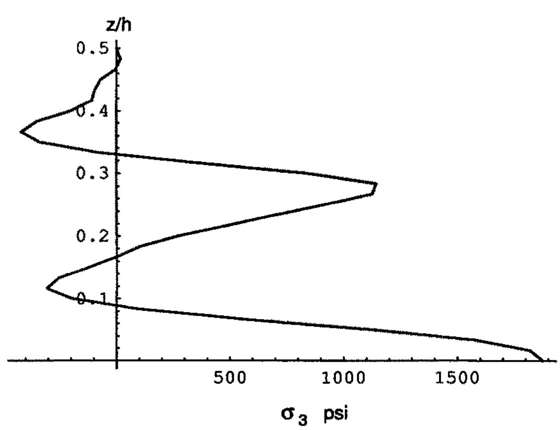

Fig. 12. Average σ_3 in $[45/-45/90/0]_{2S}$ laminate resulting from residual thermal stresses, assuming $\Delta T = -200°F$. Average taken over interval $y/h = 0$ to $y/h = 1/16$; $n = 16$.

requirements are modest. In addition, no overflow problems or matrix ill-conditioning were detected in the course of this study. The numerical results for this paper were generated on a personal computer using the Mathematica® software (Wolfram, 1991). Because Mathematica is an interpretive language, valid computer time comparisons cannot be made with programs written in compiled languages, such as Fortran.

Although the in-plane stresses are discontinuous through the thickness of the laminate, a series approximation for the stress functions which is continuous through the ply thickness appears to be an effective way to estimate the interlaminar stress components (which are continuous).

The method is powerful enough that the stress singularities at ply interfaces appear in the results as unbounded stresses with increasing degrees-of-freedom. By using an average stress criterion to predict delamination initiation, these unbounded stresses disappear and convergence to constant values is rapid.

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